

The Born Rule from a Consistency Requirement on Hidden Measurements in Complex Hilbert Space

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We formalize the hidden measurement approach within the very general notion of an interactive probability model. We narrow down the model by assuming that the state space of a physical entity is a complex Hilbert space and introduce the principle of consistent interaction which effectively partitions the space of apparatus states. The normalized measure of the set of apparatus states that interact with a pure state giving rise to a fixed outcome is shown to be in accordance with the probability obtained using the Born rule.

KEY WORDS: Born rule; hidden measurements; interactive probability model.

1. INTRODUCTION

In Aerts D. (1986), a proposal to answer the question of the arising of probabilities in quantum mechanics is given. The author argues that probability enters quantum mechanics because of a lack of knowledge about which measurement was conducted. Let us briefly outline the scheme as presented in the article to reproduce the probabilities related to the measurement of an observable \mathcal{A} with n possible alternative (and mutually exclusive) outcomes. The n eigenvectors $\{e_1, e_2, \dots, e_n\}$ of the operator A that represents the observable \mathcal{A} with n possible outcomes, can serve as a basis for the state of the entity: $q = \sum_{i=1}^n \langle q, e_i \rangle e_i$. Orthodox quantum mechanics dictates that the probability $p_q^A(a_i)$ of finding the result a_i —one of the eigenvalues $\{a_1, a_2, \dots, a_n\}$ of the eigenvector with the same index—upon execution of the measurement that corresponds to the observable \mathcal{A} when the entity is in the state q , equals

$$p_q^A(a_i) \equiv p(\mathcal{A} = a_i | q) = |\langle q, e_i \rangle|^2$$

This means that the n -tuple

$$\kappa = (p_q^A(a_1), p_q^A(a_2), \dots, p_q^A(a_n))$$

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contains all statistical information we can derive from the entity with respect to the observable \mathcal{A} and, we can use κ as a representation of the statistical state. Because the $p_q^{\mathcal{A}}(a_i)$ are constrained by the requirement $\sum_{i=1}^n p_q^{\mathcal{A}}(a_i) = 1$, we see that the statistical state is an element of the $(n - 1)$ -simplex Δ_{n-1} in \mathbb{R}^n spanned by the canonical base vectors $e_i : \kappa = \sum_i p_q^{\mathcal{A}}(a_i)e_i$. The basic idea of the “hidden measurement approach” is to associate with each measurement m a set of sub-measurements $m(\lambda)$ such that the measurement $m(\lambda)$ consists of choosing at random one of the λ and performing the measurement $m(\lambda)$ on the entity. The measurements are to be taken classically deterministic, in the sense that their operation on a fixed state always yields the same result. This is done as follows: take λ to be an n -tuple from the $(n - 1)$ -simplex: $\lambda = (\lambda_1, \dots, \lambda_n)$, $\sum \lambda_i = 1$, $\lambda_i \geq 0$. Call C_i the convex closure of the set $\{e_1, \dots, e_{i-1}, \kappa, e_{i+1}, \dots, e_n\}$. The outcome of the measurement $m(\lambda)$ is determined by λ in the following *ad hoc* way: if $\lambda \in C_i$, then the outcome reads a_i . We will not discuss the procedure when the variable λ happens to be chosen on the boundary of one of the C_i as this is a thin subset only and as such does not contribute to the final probabilities. The probability of choosing λ in the simplex C_i is calculated by assuming a uniform and normalized density for λ over Δ_{n-1} . Hence, we obtain

$$p(\lambda \in C_i | q) = \mu(C_i) / \mu(\Delta_{n-1})$$

where μ , because of the uniform measure, is simply the $(n - 1)$ -dimensional volume of the respective simplex. This volume is proportional to both the measure of any of its $(n - 2)$ -dimensional faces and to the length of the orthogonal projection of its “height” onto this face. Hence, we can easily see that the volumes of the simplices are proportional to the projections of the statistical state κ onto the base vectors. It is a matter of straightforward determinant calculus to show that $\mu(C_i) / \mu(\Delta_{n-1}) = p_q^{\mathcal{A}}(a_i)$, and hence we have

$$p(\mathcal{A} = a_i | q) = p(\lambda \in C_i | q)$$

The result is deceptively simple and it is difficult to imagine a shorter exposition of the well-known fact that there exist hidden variable models of quantum mechanics if one restricts the latter to measurements related to a single observable. This strength is immediately also a weakness of the exposition: the state of the entity is identified with the statistical state, or the set of probabilities related to a single observable, whereas in quantum mechanics we are able—at least in principle—to apply Dirac transformations to calculate the probabilities related to *all* observables we choose to measure. It is not obvious how to transform the state in the simplex when we want to measure a different observable. Is it possible to extend the procedure and make it work in Hilbert space rather than in the simplex? In the original article such an example is indeed given, but it relates only to a two-dimensional problem. However, the two-dimensional case is in some sense a

degenerate case: the possibility of sub-measurements is excluded and the Gleason theorem applies only from dimension three or higher. The latter fact has sometimes been related to the existence of hidden variable models for measurements with only two outcomes. To counter this objection, a three-dimensional model in a real Hilbert space (Aerts *et al.*, 1997a) was constructed. However, this model is much more complicated and ad-hoc than the original model and did not give a hint as to how and if the scheme would work in complex Hilbert space. Although a model in complex Hilbert space was lacking, interesting results in other directions were obtained. For example, the question of the generality of the measure theoretic construction was dealt with in a lattice-theoretic model for an experiment with a possibly infinite number of outcomes (Coecke, 1995). The two dimensional model easily allowed for parametrization of the lack of knowledge, engaging us to study the behavior of between quantum and classical descriptions by means of statistical polytopes and the violation of the axioms of quantum logic. We refer to (Aerts *et al.*, 1997b, c; 1999) and the references found there.

The present article aims at resolving two issues. The first one, raised at the end of the 1986 paper, is how to characterize the measurements that occur in a hidden measurement scheme. Can we give a less ad-hoc description of the way a measurement selects an outcome when it interacts with a state? The second issue is concerned with the realization of such a scheme in complex Hilbert space. More precisely, we will put forward a principle that partitions the set of measurements such that the measure of the set of apparatus states that actualize a fixed outcome if in interaction with a system in a pure state, is shown to be equal to the modulus squared of the inner product of the state of the entity with the eigenstate belonging to that particular outcome.

2. LACK OF KNOWLEDGE IN AN INTERACTIVE SET-UP

We will first recast the hidden measurement idea into the more general and abstract notion of an interactive system. In essence, we assume the observer is in a state $a \in M$, and the thing he observes is in a state $q \in \Sigma$. Furthermore, we assume the existence of a rule of interaction “ i ” that gives us the outcome $x \in X$ as a result of the interaction between the two states q and a :

$$i : \Sigma \times M \rightarrow X, \quad i(q, a) = x$$

We want this model to be deterministic, hence the mapping i is a function. Furthermore, we want every possible outcome x to be the result of an interaction between an entity and a measurement apparatus, hence we also require i to be surjective. Of course, surjectivity implies the possibility that different couples (q, a) lead to the same outcome: $i^{-1}(x) = \{(q, a) \in \Sigma \times M : i(q, a) = x\}$.

Suppose that we have a lack of knowledge about the precise state of the system and apparatus. With $\mathcal{B}(\Sigma)$ (and $\mathcal{B}(M)$) the Borel field of σ -additive subsets of Σ (and M), our experiment is characterized by two probability measures: μ_Σ as a probability measure $\mathcal{B}(\Sigma) \rightarrow [0, 1]$ and μ_M as probability measure $\mathcal{B}(M) \rightarrow [0, 1]$:

$$\mathcal{P}_\Sigma = (\Sigma, \mathcal{B}(\Sigma), \mu_\Sigma)$$

$$\mathcal{P}_M = (M, \mathcal{B}(M), \mu_M)$$

The way the system and the apparatus interact is governed solely by the function i : the measures themselves are independent. To define the probability of the occurrence of an outcome, we assume i is a measurable function and as such, the interaction i becomes a random variable from $\Sigma \times M$ onto X . First we need a few definitions (all sets are assumed to be non-empty):

Definition 1. An interactive probability model is a quadruple $(\mathcal{P}_\Sigma, \mathcal{P}_M, X, i)$ with

$\mathcal{P}_\Sigma = (\Sigma, \mathcal{B}(\Sigma), \mu_\Sigma)$, a probability space of a set of entity-states Σ ,
 $\mathcal{P}_M = (M, \mathcal{B}(M), \mu_M)$, a probability space of a set of apparatus-states M ,
 a non-empty set X called the outcome space, and
 a random variable $i : \Sigma \times M \rightarrow X$, called the interaction.

Definition 2. A preparation $\pi = (\psi_q, \psi_a)$ is an ensemble of entity states $\psi_q \in \mathcal{B}(\Sigma)$ and an ensemble of apparatus states $\psi_a \in \mathcal{B}(M)$.

The odds of picking a certain system state out of the ensemble ψ_q and picking one apparatus state out of ψ_a , is determined independently by the measures μ_Σ and μ_M , respectively.

Following standard probability theory, we construct the product space $\mathcal{P}_{\Sigma \times M} = (\Sigma \times M, \mathcal{B}(\Sigma) \times \mathcal{B}(M), \rho)$. The measures μ_Σ and μ_M induce the unique product probability measure $\rho : \mathcal{B}(\Sigma) \times \mathcal{B}(M) \rightarrow [0, 1]$, such that $\rho(\psi_q, \psi_a) = \mu_\Sigma(\psi_q)\mu_M(\psi_a)$. This leads to the following definition:

Definition 3. Given an interactive probability model $(\mathcal{P}_\Sigma, \mathcal{P}_M, X, i)$ and a preparation $\pi = (\psi_q, \psi_a) \in \mathcal{B}(\Sigma) \times \mathcal{B}(M)$, the interactive probability of the occurrence of the outcome x is

$$p(x | \pi) = \frac{1}{\rho(\psi_q, \psi_a)} \int_{i^{-1}(x)} d\rho$$

We stress that this definition of the probability allows for a completely natural lack of knowledge interpretation: any arising probability in the occurrence of outcomes is a consequence of the inability to prepare identical states for either the system, the apparatus, or both. This point is crucial. If we have an irreducible

uncertainty about the way we study nature, it will be impossible to give a direct operational meaning to both $(\Sigma, \mathcal{B}(\Sigma), \mu_\Sigma)$ and $(M, \mathcal{B}(M), \mu_M)$. We have to derive $(\Sigma, \mathcal{B}(\Sigma), \mu_\Sigma)$ and $(M, \mathcal{B}(M), \mu_M)$ indirectly from the interpretation of $p(x | \pi)$ as coming from an interactive probability model $(\mathcal{P}_\Sigma, \mathcal{P}_M, X, i)$. The absence of an operational definition can then be justified on principle grounds, but only if the interactive probabilistic scheme we propose is considered plausible.

3. HIDDEN MEASUREMENTS IN HILBERT SPACE

We now turn our attention to the measurement of observables with n distinct outcomes, such as the observables related to a spin- n model or to an array of n distinct detectors in a position measurement scheme. We will assume that the state space of both the entity and the apparatus is complex Hilbert space. We start by ascribing a state vector $a \in \mathcal{H}_A$ to the measurement apparatus A and a state vector $q \in \mathcal{H}_S$ to the system S . Next we assume there exists a deterministic interaction i that decides which outcome x_k from an outcome set $X = \{x_1, x_2, \dots, x_n\}$ occurs as a result of an interaction between the states of the system and the apparatus

$$i : \mathcal{H}_S \times \mathcal{H}_A \rightarrow X, \quad i(q, a) = x$$

The state of the apparatus, having much more degrees of freedom than the system it is made to measure, lives in a much bigger Hilbert space, so it is natural to assume $\dim(\mathcal{H}_A) \gg \dim(\mathcal{H}_S)$. However, all results presented in this article follow if the density of the apparatus states is proportional to the area of the subset an n -dimensional subspace where we impose the principle of consistent interaction. In the conclusion we briefly touch upon the fact that this assumption is a necessity following from an unbiasedness of the apparatus. Hence for the purpose of the present derivation we need only assume $\dim(\mathcal{H}_A)$ equals $\dim(\mathcal{H}_S)$. Having said this, let \mathcal{H}_n denote the set of unit-norm members of an n -dimensional Hilbert space over the field of complex numbers and let q and a belong to this space. Hence i is a function: $i : \mathcal{H}_n \times \mathcal{H}_n \rightarrow X$. Next we connect states to outcomes by means of the concept of an eigenvector.

Definition 4. A set $E = \{e_1, \dots, e_n\} \subset \mathcal{H}_n$ of n orthogonal vectors is called a set of *eigenvectors* iff $\forall e_k, e_l \in E$:

$$\begin{aligned} \langle e_k, e_l \rangle &= \delta_{k,l} \\ i(e_k, a) &= x_k, \forall a \in \mathcal{H}_n \end{aligned}$$

The vectors e_k play the role of eigenstates for the observable that corresponds to the measurement being made in the sense that, if the entity happens to be in one of the states e_k , it does not matter with which apparatus state it interacts; it will always yield the same result and this result depends only on the eigenstate of the

entity. We know from quantum mechanics the vectors with the desired property ($i(e_k, a) = x_k, \forall a$) are indeed simply the eigenvectors of the self-adjoint operator corresponding to the relevant observable, but as we did not assume that a self-adjoint operator represents the measurement of an observable, we have imposed this separately. Throughout the rest of the article, indices can take natural values up to n only.

3.1. THE PRINCIPLE OF CONSISTENT INTERACTION

To determine the action of i , we first define an important set of vectors that we (in absence of a better name), call a “modulus great circle segment”:

$$\langle e_k \circ a \rangle \equiv \{c \in \mathcal{H}_n : |c_j| = \sqrt{s}|a_j|, j \neq k, \\ |c_k| = \sqrt{(1-s) + s|a_k|^2}, 0 < s < 1\}$$

So the set of vectors $\langle e_k \circ a \rangle$ are those unit vectors in \mathcal{H}_n for which the modulus of each component equals the downscaled modulus of each component of c (except for c_j) by a factor \sqrt{s} . The last remaining component c_j simply follows from the normalization requirement on c . It is easy to see that this is indeed a segment of a great circle in the positive 2^n -tant of \mathbb{R}^n , obtained by taking the modulus of each component of a vector on the unit sphere in \mathbb{C}^n , hence the name “modulus great circle segment”.

Definition 5. We say the interaction $i : \mathcal{H}_n \times \mathcal{H}_n \rightarrow X$ obeys the principle of consistent interaction (PCI) iff $\forall x_k \in X; q, a, a' \in \mathcal{H}_n, e_r, e_k \in E$:

$$i(q, a) = x_k \Rightarrow i(q, a') = x_k, \forall a' \in \langle e_r \circ a \rangle, e_k \neq e_r$$

In words, the principle of consistent interaction says that, for a fixed state q of the entity, if the interaction with an apparatus state a gives rise to an outcome x_k , then so does the interaction with any other apparatus state a' that belongs to the “modulus great circle segments” between the apparatus state a and any eigenvector belonging to another outcome than x_k . We will first discuss some mathematical consequences of this principle and postpone a possible interpretation to the concluding section of this paper. The sets $\langle e_r \circ a \rangle$ constitute only a thin subset of the state space \mathcal{H}_n of the apparatus. Nevertheless, it is evident that the principle poses a severe constraint on the set of possible partitions of this space. To see just how constraining the PCI is, let us investigate it by means of the component-wise product of a complex vector with its complex conjugate that sends elements of the complex unit-sphere $S_n = \{z \in \mathbb{C}^n : \sum_{i=1}^n z_i z_i^* = 1\}$ onto the $(n - 1)$ -simplex $\Delta_{n-1} = \{x \in \mathbb{R}_+^n : \sum_{i=1}^n x_i = 1\}$:

$$\tau : S_n \rightarrow \Delta_{n-1}$$

$$\tau(z) = (z_1 z_1^*, z_2 z_2^*, \dots, z_n z_n^*)$$

Let us translate the PCI to the simplex by means of τ . A simple calculation shows that $\tau(\langle e_r \circ a \rangle) =]\tau(e_r), \tau(a)[$, that is, τ maps “modulus great circle segments” to open line-segments in Δ_{n-1} . The translation of the PCI to the simplex Δ_{n-1} then reads:

$$i(\tau(q), \tau(a)) = x_k \Rightarrow i(\tau(q), \tau(a')) = x_k, \forall \tau(a') \in]\tau(e_r), \tau(a)[, \tau(e_k) \neq \tau(e_r)$$

It is not difficult to define a partition in the simplex that is consistent with the PCI. We denote by $]A[$, the relative interior of the convex closure of A , and define (with slight abuse of the notation C_k^q rather than $C_k^{\tau(q)}$) the sets

$$C_k^q =]x_1, \dots, x_{k-1}, \tau(q), x_{k+1}, \dots, x_n[$$

This division of Δ_{n-1} into separate sets C_k^q takes the form of a special type of triangulation, which is a simple generalization of a barycentric division, and is in fact affinely isomorphic to it. We have encountered this particular partition in the Introduction. Just as was the case there, assume now that the interaction i in the simplex is defined as follows:

$$\tau(a) \in C_k^q \Rightarrow i(\tau(q), \tau(a)) = x_k$$

It is easy to see that for every mapped apparatus state $\tau(a) \in C_k^q$ we indeed have that $] \tau(e_k), \tau(a) [\subset C_k^q$. Hence, elements of $] \tau(e_k), \tau(a) [$ also give rise to an outcome x_k , and as such are in accordance with the PCI. Likewise, we can see that for every $a \in \tau^{-1}(C_k^q)$, we have that $\langle e_r \circ a \rangle \subset \tau^{-1}(C_k^q)$ leading to the same conclusion. One can easily convince oneself intuitively that no other partition of the set of apparatus states can satisfy the PCI, and refer the interested reader to (Aerts, 2002), where a full proof, utilising mainly elementary convex geometry, can be found. Note that $\cup_k \tau^{-1}(C_k^q) = \mathcal{H}_n \setminus M_0$, where $M_0 = \cup_k \partial[\tau^{-1}(C_k^q)]^2$ is the set of boundaries of the closure of the sets $\tau^{-1}(C_k^q)$. Clearly M_0 is a null set with respect to an n -measure. Hence, for probabilistic purposes, the $\tau^{-1}(C_k^q), k = 1, \dots, n$ constitute what one might call an “effective partition” or a “partition modulo null-sets” of the complex unit sphere.

4. THE BORN RULE

What constitutes a good measurement? Well, to be sure, a measurement setup is supposed to give maximal information about the state of the entity it is observing and minimal information about the state of the apparatus. For the probability space \mathcal{P}_Σ related to the entity, this means that μ_Σ becomes a point measure and hence

² It was pointed out to me by T. Durt that the set M_0 and the set of points of unstable equilibrium in the Bohm–Bub hidden variable model (Bohm and Bub, 1966) coincide, showing there is a definite and close relation between the two approaches.

the ensemble ψ_q reduces to a singleton. By a well-known theorem in information theory we have that, minimization of the information in the probability space \mathcal{P}_M related to the apparatus, μ_M becomes a uniform measure and the ensemble ψ_a the whole Hilbert space \mathcal{H}_n . It turns out that under these two assumptions, together with the PCI, we recover the Born rule.

Theorem 1. *Given an Interactive Probability Model $(\mathcal{P}_{\mathcal{H}_n}, \mathcal{P}_{\mathcal{H}_n}, X, i)$ in complex Hilbert space with i satisfying the PCI. Assume the preparation $\mathcal{B}(\Sigma) \times \mathcal{B}(M) \ni \pi = (q, \mathcal{H}_n)$ where q is a singleton. With $\{e_1, \dots, e_n\}$ a set of eigenvectors and e_k the eigenvector corresponding to the outcome $x_k \in X$, we have:*

$$p(x_k | \pi) = |\langle q, e_k \rangle|^2$$

Proof: We start with the definition of the interactive probability under the assumptions of the theorem

$$\begin{aligned} p(x_k | \pi) &= \frac{1}{\rho(q, \mathcal{H}_n)} \int_{i^{-1}(x_k)} d\rho \\ &= \frac{\mu_\Sigma(q)}{\mu_\Sigma(q)\mu_M(\mathcal{H}_n)} \int_{\tau^{-1}(C_k^q)} d\mu_M \\ &= \frac{\mu_M(\tau^{-1}(C_k^q))}{\mu_M(\mathcal{H}_n)} \end{aligned}$$

This last equation simply tells us that the probability of getting the outcome x_k equals the ratio of the apparatus states that result in that outcome to the total of all possible apparatus states. The calculation of the quantity $\mu_M(\tau^{-1}(C_k^q))$ is greatly facilitated by realizing τ preserves probability measures. In virtue of a lemma presented after this argument, the last expression becomes

$$\frac{v(C_k^q)}{v(\Delta_{n-1})}$$

The calculation of this last quantity was outlined in the introduction of this article and demonstrated explicitly in (Aerts, 1986).

$$\begin{aligned} &= \tau(q) \cdot \tau(e_k) = q_k q_k^* \\ &= |\langle q, e_k \rangle|^2 \end{aligned}$$

□

Lemma 1. Let $(\Delta_{n-1}, \mathcal{B}(\Delta_{n-1}), \mu)$ and $(S_n, \mathcal{B}(S_n), \nu)$ be the two measure spaces. Then for $A \in \mathcal{B}(\Delta_{n-1})$ and $\tau^{-1}(A) \in \mathcal{B}(S_n)$, we have

$$\nu(\tau^{-1}(A)) = \frac{2\pi^n}{\sqrt{n}} \mu(A)$$

Proof: Let A be an arbitrary open convex set in $\Delta_1: A = \{(x_1, x_2) : a < x_1 < b, x_2 = 1 - x_1\}$. Evidently, $\mu(A) = \sqrt{2}(b - a)$. Let B be the pull-back of A under τ as

$$B = \{(z_1, z_2) \in Z_1 \times Z_2 \subset \mathbb{C}^2 : Z_1 = \{z_1 : a < |z_1|^2 < b\}, \\ Z_2 = \{z_2 : z_2 = \sqrt{1 - |z_1|^2} e^{i\theta}, \theta \in [0, 2\pi]\}$$

Clearly,

$$\nu(B) = \nu(Z_1)\nu(Z_2) = \pi(b - a)2\pi = \frac{2\pi^2}{\sqrt{2}} \mu(A)$$

□

Hence the theorem holds for convex sets if $n = 2$. This conclusion can readily be extended to an arbitrary $(n - 1)$ -dimensional rectangle set A in Δ_{n-1} :

$$A = \left\{ \left(x_1, \dots, x_{n-1}, 1 - \sum_{i=1}^{n-1} x_i \right) : \forall i = 1, \dots, n - 1 : a_i < x_i < b_i; a_i, b_i \in [0, 1] \right\}$$

Its measure factorizes into

$$\mu(A) = \sqrt{n} \prod_{i=1}^{n-1} (b_i - a_i)$$

Next, consider n -tuples of complex numbers

$$B = \{(z_1, z_2, \dots, z_n) \in Z_1 \times \dots \times Z_n\} \\ Z_i = \{z_i \in \mathbb{C} : a_i < |z_i|^2 < b_i, i \neq n\}, \\ z_n = \sqrt{1 - |z_1|^2 - \dots - |z_{n-1}|^2} e^{i\theta_n}, \theta_n \in [0, 2\pi]\}$$

Clearly $\tau(B) = A$. The measure of B can be factorized as

$$\nu(B) = \nu(Z_1)\nu(Z_2), \dots, \nu(Z_n) \\ = 2\pi \prod_{i=1}^{n-1} \pi(b_i - a_i) = \frac{2\pi^n}{\sqrt{n}} \mu(A)$$

Hence the theorem holds for an arbitrary rectangleset A . But every open set in Δ_{n-1} can be written as a pair-wise disjoint countable union of rectangular sets. It follows that $\nu(\tau^{-1}(\cdot)) = \frac{2\pi^n}{\sqrt{n}}\mu(\cdot)$ for all open sets in Δ_{n-1} . Both ν and μ are finite Borel measures because Δ_{n-1} and S_n are both compact subsets of a vectorspace of countable dimension. Therefore, they must be regular measures. But a regular measure is completely defined by its behavior on open sets. Hence the theorem holds for Borel sets. \square

5. CONCLUDING REMARKS

Besides the fact that the PCI defines the partition of the set of apparatus states, it is also interpretable as some sort of “proposal-consistent-answer-game”. To see this, make the comparison with the well-known game of “warm” and “cold”. The object of the game is to guess the location of an unknown object in a room, using the clues “warmer” and “colder” given by someone who knows the location of the object. The equivalent of the PCI for this game would be that if the guesser in his next guess is *further* from the object than a former guess, his reply has to be “colder”. Imagine now a straightforward multi-dimensional generalization of the game played in the $(n - 1)$ -simplex, and with as possible answers the n vertices of the simplex. The state vector a , then represents the measurement apparatus and is a “proposal” both *to* the state and *for* the state, as if the measurement asks the question: can you give me a clue about your true location if my guess would be that it is somewhere here you are residing? Now the entity, in response to that proposal has to give a hint about its true location by giving the unique outcome that is in accordance with the PCI. This answer can only be one of the n outcomes corresponding to the eigenvectors and, seen from the point of view of the guesser (the apparatus), it gives n alternative directions to choose from. The PCI does not tell what happens when the next guess is closer to the eigenvector corresponding to the outcome given to the former guess. It doesn’t need to. What the PCI requires, is that the response of the entity is such that if the answer was “vertex x_i ” and the guesser chooses to ignore that directional hint and places his guess closer in the direction of another vertex (rather than closer to vertex x_i) the answer will still have to be x_i . Once you think of it this way, the PCI indeed expresses a very basic form of consistency, and it is nice to see that this condition alone partitions the set of apparatus states. As such the PCI seems to show a relationship between the geometry of Hilbert space and the probabilistic inferences made therein. The model we propose in the case of quantum mechanics differs from the game of warm and cold, in that, each new interaction forgets the outcome that was given in response to the interaction with a former apparatus state. This is connected to the minimization of the information regarding the interactional part of the apparatus state and the corresponding uniform density of states. It is also essentially a matter

of the apparatus being unbiased: the apparatus should not be more sensitive to some states than to others, nor should it know in advance which entity state is going to be presented to it. Together, these assumptions led us to the Born rule. Mathematically speaking, Gleason's theorem gives you more for less, apparently rendering the result redundant. However, we have gained an interpretation. If there is in nature something like "the observer and the observed together producing the phenomenon," then we believe the scheme outlined here is sensible, and, as we hope to have shown, not too difficult to translate to complex Hilbert space to recover the Born rule by an integration over unknown observer states, in accordance with the original hidden measurement proposal.

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